Gaussian Particle Implementations of Probability Hypothesis Density Filters

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Abstract— The Probability Hypothesis Density (PHD) filter is a multiple-target filter for recursively estimating the number of targets and their state vectors from sets of observations. The filter is able to operate in environments with false alarms and missed detections. Two distinct algorithmic implementations of this technique have been developed. The first of which, called the Particle PHD filter, requires clustering techniques to provide target state estimates which can lead to inaccurate estimates and is computationally expensive. The second algorithm, called the Gaussian Mixture PHD (GM-PHD) filter does not require clustering algorithms but is restricted to linear-Gaussian target dynamics, since it uses the Kalman filter to estimate the means and covariances of the Gaussians. Extensions for the GM-PHD filter allow for mildly non-linear dynamics using extended and Unscented Kalman filters. A new particle implementation of the PHD filter which does not require clustering to determine target states is presented here. The PHD is approximated by a mixture of Gaussians, as in the GM-PHD filter but the transition density and likelihood function can be non-linear. The resulting filter no longer has a closed form solution so Monte Carlo integration is applied for approximating the prediction and update distributions. This is calculated using a bank of Gaussian particle filters, similar to the procedure used with the Gaussian sum particle filter. The new algorithm is derived here and presented with simulated results.

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1. INTRODUCTION

The mathematical theory of filtering is concerned with estimating the state of a stochastic process recursively based on the history of partial observations. A natural application of filtering theory is the problem of target tracking, where the state of a target, such as its position, velocity or other features of interest is estimated based on detections observed from a sensor. If the state transition and observation functions are linear and the process and observation noises are Gaussians, then an optimal solution can be computed analytically in the form of a Kalman filter [1]. Non-linear extensions to the Kalman filter include the extended Kalman filter [2] and Julier's unscented Kalman filter [3]. Alternative filtering techniques, which allow for more general target dynamics, include the Gaussian sum filter [4] and the particle filter [5].

When the problem is extended to a multiple-target scenario where the sensor observed multiple noisy observations with false alarms and missed detections, the problem becomes much more complex. Most approaches to this problem involve associating measurements to individual filters using data association techniques such as Nearest Neighbour, Joint Probabilistic Data Association and Multiple Hypothesis Tracking [6]. These techniques can be difficult to implement and the framework does not allow for any verifiable mathematical notion of optimality.

Mahler's Finite set statistics (FISST) provides a general systematic foundation for multi-target filtering based on the theory of random finite set (RFS). The basic idea is to do filtering on set-valued observations and set-valued states. One such technique, known as the Probability Hypothesis Density (PHD) filter [7], propagates the posterior intensity function, or first-order statistical moment, of the random finite set of targets in time. This approach has led to efficient multiple target tracking algorithms for jointly estimating the number of targets and their states from a sequence of observation sets with data association uncertainty, missed detections, false alarms and noisy measurements. The first such algorithm, known as the particle PHD filter [8], used a sequential Monte Carlo approach for approximating the posterior intensity function. The target states can be determined from the multi-modal empirical density by using clustering techniques, such as the k-means or Expectation-Maximization algorithms, which can lead to inaccurate target state estimates [9]. The second algorithm, known as the Gaussian Mixture PHD filter, provided a closed-form solution to the

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PHD recursion under linear-Gaussian conditions by approximating the intensity function by a weighted mixture of Gaussians [10].

The Gaussian Mixture PHD (GM-PHD) filter provides an analytic solution to the PHD filter [10]. The multiple-target state is represented by a mixture of Gaussians. Target states can be determined more easily with this algorithm than the Particle PHD filter, by taking the components in the mixture with the largest weights. In addition, the trajectories of the targets can be estimated by considering the evolution of individual Gaussians within the mixture [11]. Furthermore, this filter can be generalised to linear Jump Markov Models for switching between multiple models [12]. Non-linear extensions to this algorithm include the extended Kalman PHD (EK-PHD) filter and unscented Kalman (UK-PHD) filter [10], although these can be divergent in highly non-linear problems.

This paper presents a new particle implementation of the PHD filter based on the GM-PHD filter. Clustering techniques are not required to determine the target states and the Gaussian particle approximation is asymptotically consistent even for non-linear target models. The PHD is approximated by a sum of Gaussians, as is the case in the GM-PHD filter, but the target dynamics and observation can be non-linear. The resulting filter is no longer has a closed form solution so Monte Carlo integration is applied for approximating the prediction and posterior densities. This is calculated using a bank of Gaussian particle filters [13], similar to the procedure used with the Gaussian sum particle filter [14]. At each stage of the algorithm, the mixture is represented by a weighted sum of Gaussians. The methods used for estimating the target states and their trajectories from the GM-PHD filter are directly applicable. We also extend our technique to the newly developed cardinalized PHD (CPHD) filter, which generalizes the PHD recursion by jointly propagating the posterior intensity function and the posterior distribution of the number of targets [15].

2. Random Set Filtering

The multiple target tracking framework based on random-sets was first proposed by Mahler [7] as a mathematically rigorous attempt to unify the problems of detection, classification and tracking. The approach uses a Bayesian paradigm for recursively estimating and updating a multi-target density function based on measurements received at each time-step.

Multiple-target filtering requires the unobserved signal process $X_{0:k} = (X_0, ..., X_k)$ to be estimated based on the σ-algebra generated by the sets of observations up to time k, $Z_{1:k} := \sigma((Z_1, ..., Z_k))$, i.e. to obtain $\hat{X}_k = \{\hat{x}_{k,1}, ..., \hat{x}_{k,\hat{T}_k}\}$, where $\hat{x}_{k,i}$ are the individual target estimates and $\hat{T}_k$ is the estimate of the number of targets at time k. This is done by recursively calculating the posterior distribution, or filtering distribution, $p_k(\cdot|Z_{1:k})$. Note that this differs from the usual approach of managing single-target filters with ad-hoc data association techniques in that it provides a coherent mathematical framework.

The set of objects tracked at time k is modelled by the point process or Random Finite Set (RFS)

$$X_k = \left(\bigcup_{x \in X_{k-1}} S_{k|k-1}(x) \right) \cup \left(\bigcup_{x \in X_{k-1}} B_{k|k-1}(x) \right) \cup \Gamma_k. \quad (1)$$

where $S_{k|k-1}$ is the RFS of targets survived at time t from multi-target state X_{k-1} [1] at time $k-1$, $B_{k|k-1}$ is the RFS of targets spawned from X_{k-1} and Γ_k is the RFS of targets that appear spontaneously at time t. The multi-target measurement at time t is modelled by RFS

$$Z_k = K_k \cup \left(\bigcup_{x \in X_k} \Theta_k(x) \right), \quad (2)$$

where $\Theta_k(X_k)$ is the RFS of measurements from multi-target state X_k and K_k is the RFS of measurements due to clutter.

The optimal multi-target Bayes filter propagates the multi-target posterior density $p_k(\cdot|Z_{1:k})$ conditioned on the sets of observations up to time k, $Z_{1:k}$, with the following recursion

$$p_{k|k-1}(X_k|Z_{1:k-1}) = \int f_{k|k-1}(X_k|X) p_{k-1}(X|Z_{1:k}) \mu_s(dX), \quad (3)$$

$$p_k(X_k|Z_{1:k}) = \frac{g_k(Z_k|X_k) p_{k|k-1}(X_k|Z_{1:k-1})}{\int g_k(Z_k|X) p_{k|k-1}(X|Z_{1:k-1}) \mu_s(dX)}, \quad (4)$$

where the dynamic model is governed by the transition density $f_{k|k-1}(X_k|X_{k-1})$ and multi-target likelihood $g_k(Z_k|X_k)$ and μ_s takes the place of the Lebesgue measure, as described in [8].

The function $g_{k|k}(Z_k|X_k)$ is the joint multi-target likelihood function, or global density, of observing the set of measurements, Z, given the set of target states, X, which is the total probability density of association between measurements in Z and parameters in X. The parameters for this density are the set of observations, $Z = \{z_1, ..., z_k\}$, the unknown set of target states, $X = \{x_1, ..., x_k\}$, the sensor noise distribution, or observation noise, clutter models, and the detection profile of sensor or field of view (FoV).

The computational complexity of the joint multi-target likelihood grows exponentially with the number of targets and so becomes numerically intractible [16]. The Probability Hypothesis Density (PHD) filter was derived to provide a tractable sub-optimal strategy for determining the set of target states at each iteration by using the first-order statistical moment of the multi-target posterior distribution [7]. The PHD filter is a recursion which propagates the first-order moment

[1] Note that the same notation is used for a random variable and its realization.

of the multi-target posterior which is known in the tracking community as the PHD [7]. To ensure that a closed-form recursion is possible, a Poisson assumption is made after the prediction and update steps. Sets of target estimates are determined by obtaining peaks of the PHD, since this property represents the expectation. Alternatively, the random-sets can be viewed as point processes and the prediction is computed via Markov shifts [17]. In this case, the PHD can be considered as an intensity function.

The Probability Hypothesis Density (PHD) filter

The PHD filter is an approximation developed to alleviate the computational intractability in the multi-target Bayes filter. Instead of propagating the multi-target posterior density in time, the PHD filter propagates the posterior intensity, a first-order statistical moment of the posterior multi-target state [7]. This strategy is reminiscent of the constant gain Kalman filter, which propagates the first moment (the mean) of the single-target state.

For a RFS X on $\mathcal{X}$ with probability distribution P, its first-order moment is a non-negative function v on $\mathcal{X}$, called the *intensity*, such that for each region $S \subseteq \mathcal{X}$ [18]

$$\int |X \cap S| P(dX) = \int_S v(x)dx. \tag{5}$$

In other words, the integral of v over any region S gives the expected number of elements of X that are in S. Hence, the total mass $\hat{N} = \int v(x)dx$ gives the expected number of elements of X. The local maxima of the intensity v are points in $\mathcal{X}$ with the highest local concentration of expected number of elements, and hence can be used to generate estimates for the elements of X. The simplest approach is to round $\hat{N}$ and choose the resulting number of highest peaks from the intensity. The intensity is also known in the tracking literature as the Probability Hypothesis Density (PHD) [7].

An important class of RFSs, namely the Poisson RFSs, are those completely characterized by their intensities. A RFS X is *Poisson* if the cardinality distribution of X, $\Pr(|X| = n)$, is Poisson with mean $\hat{N}$, and for any finite cardinality, the elements x of X are independently and identically distributed according to the probability density $v(\cdot)/\hat{N}$.

Consider the following assumptions:

A.1. Each target evolves and generates observations independently of one another,

A.2. Clutter is independent of target-originated measurements,

A.3. The predicted multi-target RFS governed by $p_{k|k-1}$ is Poisson.

Remark 1. Assumptions A.1 and A.2 are standard in most tracking applications. The additional assumption A.3 is a reasonable approximation in applications where interactions between targets are negligible [19]. In fact, A.3 is completely satisfied when X_{k-1}, Γ_k are Poisson, and there is no spawning.

Let v_k and $v_{k|k-1}$ denote the respective intensities associated with the multi-target posterior density p_k and the multi-target predicted density $p_{k|k-1}$ in the recursion (3)-(4). Under assumptions A.1-A.3, it can be shown (using FISST [19] or classical probabilistic tools [17]) that the posterior intensity can be propagated in time via the PHD recursion. The prediction equation is given by

$$v_{k|k-1}(x) = \int p_{S,k}(\zeta) f_{k|k-1}(x|\zeta) v_{k-1}(\zeta) d\zeta + \gamma_k(x), \tag{6}$$

where

$$\begin{aligned} f_{k|k-1}(\cdot|\zeta) &= \text{single target transition density at time } k, \\ p_{S,k}(\zeta) &= \text{probability of target existence at time } k, \\ \gamma_k(\cdot) &= \text{intensity of spontaneous births at time } k, \end{aligned}$$

and the update equation is given by

$$\begin{aligned} v_k(x) &= [1 - p_{D,k}(x)] v_{k|k-1}(x) \\ &+ \sum_{z \in Z_k} \frac{p_{D,k}(x) g_k(z|x) v_{k|k-1}(x)}{\kappa_k(z) + \int p_{D,k}(\xi) g_k(z|\xi) v_{k|k-1}(\xi) d\xi} \end{aligned} \tag{7}$$

where

$$\begin{aligned} Z_k &= \text{measurement set at time } k, \\ g_k(\cdot|x) &= \text{single target measurement likelihood at time} k \\ p_{D,k}(x) &= \text{probability of target detection at time } k \\ \kappa_k(\cdot) &= \text{intensity of clutter measurements at time } k, \end{aligned}$$

For simplicity we do not consider spawning in this paper. It is clear from (6)-(7) that the PHD filter completely avoids the combinatorial problem that arises from the unknown association of measurements with appropriate targets. Furthermore, since the posterior intensity is a function on the single-target state space $\mathcal{X}$, the PHD recursion requires much less computational power than the multi-target recursion (3)-(4), which operates on $\mathcal{F}(\mathcal{X})$.

The Cardinalized PHD Recursion

The cardinalized PHD (CPHD) recursion is a higher order generalization of the PHD recursion proposed by Mahler in [15, 20]. The CPHD recursion jointly propagate the intensity function and the cardinality distribution (the probability distribution of the number of targets). The cardinality information improves the accuracy and stability of the cardinality estimates, which in turn improves the target state estimates.

Similar to the PHD recursion, the CPHD recursion rests on the following assumptions regarding the target dynamics and observations:

- Each target evolves and generates measurements independently of one another;

• The birth RFS and the surviving RFSs are independent of each other;
• The clutter RFS is an i.i.d cluster process and independent of the measurement RFSs;
• The prior and predicted multi-target RFSs are i.i.d cluster processes.

The following notation is needed in the CPHD recursion. We denote by C_j^ℓ the binomial coefficient $\frac{\ell!}{j!(\ell-j)!}$, P_j^n the permutation coefficient $\frac{n!}{(n-j)!}$, $\langle\cdot,\cdot\rangle$ the inner product defined between two real valued functions α and β by $\langle\alpha,\beta\rangle = \int\alpha(x)\beta(x)dx$ (or $\sum_{\ell=0}^{\infty}\alpha(\ell)\beta(\ell)$ when α and β are real sequences), and $e_j(\cdot)$ the elementary symmetric function of order j defined for a finite set Z of real numbers by $e_j(Z) = \sum_{S\subseteq Z,|S|=j}\prod_{\zeta\in S}\zeta$ with $e_0(Z) = 1$ by convention.

Let $p_{k|k-1}$ denote cardinality distribution associated with the predicted multi-target state, and p_k denote the intensity and cardinality distribution associated with the posterior multi-target state. Then, the multi-target posterior intensity and posterior cardinality distribution can be propagated in time with the *CPHD prediction* and *CPHD update* [15,20–22]

The CPHD prediction is given by (8)-(9):

$$p_{k|k-1}(n) = \sum_{j=0}^{n} p_{\Gamma,k}(n-j)\Pi_{k|k-1}[v_{k-1},p_{k-1}](j), \quad (8)$$

$$v_{k|k-1}(x) = \int p_{S,k}(\zeta)f_{k|k-1}(x|\zeta)v_{k-1}(\zeta)d\zeta+\gamma_k(x), \quad (9)$$

where

$$\Pi_{k|k-1}[v,p](j) = \sum_{\ell=j}^{\infty} C_j^\ell \frac{\langle p_{S,k},v\rangle^j\langle 1-p_{S,k},v\rangle^{\ell-j}}{\langle 1,v\rangle^\ell}p(\ell), \quad (10)$$

$$p_{\Gamma,k}(\cdot) = \text{cardinality distribution of births at time } k. \quad (11)$$

The CPHD update is given by (12)-(13):

$$p_k(n) = \frac{\Upsilon_k^0[v_{k|k-1};Z_k](n)p_{k|k-1}(n)}{\langle\Upsilon_k^0[v_{k|k-1};Z_k],p_{k|k-1}\rangle}, \quad (12)$$

$$v_k(x) = (1-p_{D,k}(x))\frac{\langle\Upsilon_k^1[v_{k|k-1};Z_k],p_{k|k-1}\rangle}{\langle\Upsilon_k^0[v_{k|k-1};Z_k],p_{k|k-1}\rangle}v_{k|k-1}(x) \quad (13)$$

$$+\sum_{z\in Z_k}\psi_{k,z}(x)\frac{\langle\Upsilon_k^1[v_{k|k-1};Z_k\backslash\{z\}],p_{k|k-1}\rangle}{\langle\Upsilon_k^0[v_{k|k-1};Z_k],p_{k|k-1}\rangle}v_{k|k-1}(x),$$

where

$$\Upsilon_k^u[v;Z](n) = \sum_{j=0}^{\min(|Z|,n)}(|Z|-j)!p_{K,k}(|Z|-j)P_{j+u}^n \quad (14)$$

$$\times\frac{\langle 1-p_{D,k},v\rangle^{n-(j+u)}}{\langle 1,v\rangle^n}e_j\left(\Xi_k(v,Z)\right),$$

$$\psi_{k,z}(x) = \frac{\langle 1,\kappa_k\rangle}{\kappa_k(z)}g_k(z|x)p_{D,k}(x), \quad (15)$$

$$\Xi_k(v,Z) = \left\{\langle v,\psi_{k,z}\rangle : z\in Z\right\}, \quad (16)$$

$$p_{K,k}(\cdot) = \text{cardinality distribution of clutter at time } k. \quad (17)$$

The above form of the CPHD recursions (see [21, 22]) facilitates the presentation of our new implementations. This form is equivalent to the original CPHD recursions derived in [15,20].

3. Gaussian Mixture Particle PHD Filters

In this section we present the main result of our paper, a Gaussian particle implementation of the PHD filters. We first summarize the key elements of the Gaussian mixture PHD filter to facilitate the presentation of our results. The Gaussian mixture particle implementation of the PHD recursion is then presented, followed by the extension to the CPHD filter.

The Gaussian Mixture PHD filter

The Gaussian Mixture PHD filter provides a closed-form solution to the PHD filter [10] under linear assumptions on the system and observation equations with Gaussian process and observation noises. The intensity function is approximated at each stage with a weighted sum of Gaussians. More specifically, under the assumption that the posterior intensity at time $k-1$ is a Gaussian mixture and that the motion model is linear with Gaussian process noise, it is shown that the predicted intensity to time k is a Gaussian mixture. Furthermore, if the predicted intensity to time k is a Gaussian mixture, under the assumption that the observation model is linear and that the measurement noise is Gaussian, then the updated intensity is also a Gaussian mixture. The constituent means and covariances of the Gaussian mixture posterior intensity can be easily computed with the Kalman recursion (see [10] for details). Some non-linear extensions can be accommodated by replacing the Kalman filter equations by the extended Kalman filter [2] or unscented Kalman filter [3]. Convergence of the approximation errors in the GM-PHD filter has been established by Clark and Vo [23], based on a consequence of Wiener's theorem on approximation that any density function can be represented by a mixture of Gaussians [24], and convergence results for the particle PHD filter [25].

The multiple target states in the GM-PHD filter are estimated by taking the Gaussian components with highest weights. This leads naturally to an extension of the GM-PHD filter which allows the evolution of individual target states to be determined over time, which ensures the continuity of the target tracks [11]. The performance of this extension compared favourably to the track-oriented MHT filter for tracking multiple targets in clutter. A demonstration of the GM-PHD filter with track continuity has been compared with the particle PHD filter for tracking in forward-scan sonar [26], where it was shown that it performed better in higher levels of clutter and is computationally much faster.

Closed-form solutions to the PHD recursion require, in addition to assumptions A.1-A.3, a linear Gaussian multi-target model. Along with the standard linear Gaussian model for individual targets, the linear Gaussian multi-target model includes certain assumptions on the birth, death and detection of targets. These are summarized below:

A.4. Each target follows a linear Gaussian dynamical and observation model i.e.

$$f_{k|k-1}(x|\zeta) = \mathcal{N}(x; F_{k-1}\zeta, Q_{k-1}), \tag{18}$$
$$g_k(z|x) = \mathcal{N}(z; H_k x, R_k), \tag{19}$$

where $\mathcal{N}(\cdot; m, P)$ denotes a Gaussian density with mean m and covariance P, F_{k-1} is the state transition matrix, Q_{k-1} is the process noise covariance, H_k is the observation matrix, and R_k is the observation noise covariance.

A.5. The survival and detection probabilities are state independent, i.e.

$$p_{S,k}(x) = p_{S,k}, \tag{20}$$
$$p_{D,k}(x) = p_{D,k}. \tag{21}$$

A.6. The intensities of the birth RFSs is a Gaussian mixture of the form

$$\gamma_k(x) = \sum_{i=1}^{J_{\gamma,k}} w_{\gamma,k}^{(i)} \mathcal{N}(x; m_{\gamma,k}^{(i)}, P_{\gamma,k}^{(i)}), \tag{22}$$

where $J_{\gamma,k}$, $w_{\gamma,k}^{(i)}$, $m_{\gamma,k}^{(i)}$, $P_{\gamma,k}^{(i)}$, $i = 1, \ldots, J_{\gamma,k}$, are given model parameters that determine the shape of the birth intensity.

The following two properties of Gaussian distributions are used to derive the Kalman filter, and will be used for the derivations in the next section. These properties are taken from the Kalman filter derivation by Ho and Lee [27] and were used in the derivation of the GM-PHD filter [10]

$$\int \mathcal{N}(x; F_{k-1}\zeta, Q_{k-1}) \mathcal{N}(\zeta; m_{k-1}, P_{k-1}) d\zeta = \mathcal{N}(x; F_{k-1} m_{k|k-1}, P_{k|k-1}) \tag{23}$$

$$\mathcal{N}(z; H_k x, R_k) \mathcal{N}(x; m_{k|k-1}, P_{k|k-1}) = \mathcal{N}(z; \hat{z}_{k|k-1}, S_{k|k-1}) \mathcal{N}(x; m_k(z), P_{k|k}) \tag{24}$$

We now present the Gaussian Mixture PHD recursion.
GM-PHD Prediction: Suppose that the posterior intensity at time $k-1$ is a Gaussian mixture of the form

$$v_{k-1}(x) = \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)} \mathcal{N}(x; m_{k-1}^{(i)}, P_{k-1}^{(i)}). \tag{25}$$

Substituting linear transition density $f_{k|k-1}$, posterior intensity v_{k-1} and and birth intensity γ_k into the PHD prediction, we have

$$\begin{aligned} &v_{k|k-1}(x) \\ &= p_{S,k} \int \mathcal{N}(x; F_{k-1}\xi, Q_{k-1}) \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)} \mathcal{N}(\xi; m_{k-1}^{(i)}, P_{k-1}^{(i)}) d\xi \\ &+ \gamma_k(x) \qquad (26) \\ &= p_{S,k} \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)} \int \mathcal{N}(x; F_{k-1}\xi, Q_{k-1}) \mathcal{N}(\xi; m_{k-1}^{(i)}, P_{k-1}^{(i)}) d\xi \\ &+ \gamma_k(x) \qquad (27) \\ &= p_{S,k} \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)} \mathcal{N}(x; m_{k|k-1}^{(i)}, P_{k|k-1}^{(i)}) + \sum_{i=1}^{J_{\gamma,k}} w_{\gamma,k}^{(i)} \mathcal{N}(x; m_{\gamma,k}^{(i)}, P_{\gamma,k}^{(i)}). \qquad (28) \end{aligned}$$

where the means $m_{k|k-1}^{(i)}$ and covariances $P_{k|k-1}^{(i)}$ of each component in the Gaussian mixture are given by the Kalman prediction

$$m_{k|k-1}^{(i)} = F_{k-1} m_{k-1}^{(i)}, \tag{29}$$
$$P_{k|k-1}^{(i)} = Q_{k-1} + F_{k-1} P_{k-1}^{(i)} F_{k-1}^T,. \tag{30}$$

The equality (28) comes from a property of Gaussian distributions in equation (23). When the transition function is mildly non-linear, these can be replaced with the extended or unscented Kalman filter prediction equations. We will show that this can also be computed using Monte Carlo integration. In the original version of the Gaussian mixture PHD filter, the means, $m_{S,k|k-1}^{(j)}$, and covariances, $P_{S,k|k-1}^{(j)}$, were calculated using the Kalman, extended Kalman or unscented Kalman filter prediction. We will show that, for non-linear target dynamics where there is no closed form solution or linearised approximation, we can use a Monte Carlo approximation.

GM-PHD Update: Suppose that the predicted intensity to time k is a Gaussian mixture of the form

$$v_{k|k-1}(x) = \sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} \mathcal{N}(x; m_{k|k-1}^{(i)}, P_{k|k-1}^{(i)}). \tag{31}$$

Substituting the prediction intensity $v_{k|k-1}$ and linear Gaussian observation g_k into the PHD update equation, and using the properties of Gaussians (23), (24), this simplifies to

$$v_k(x) = (1 - p_{D,k}) v_{k|k-1}(x) + \sum_{z \in Z_k} \sum_{j=1}^{J_{k|k-1}} w_k^{(j)}(z) \mathcal{N}(x; m_{k|k}^{(j)}(z), P_{k|k}^{(j)}) \tag{32}$$

where the weights of the components are given by (33) and the means and covariances are calculated with the Kalman update equations (34) and (35)

$$w_k^{(j)}(z) = \frac{p_{D,k} w_{k|k-1}^{(j)} \mathcal{N}(z; \hat{z}_{k|k-1}^{(j)}, S_{k|k-1}^{(j)})}{\kappa_k(z) + p_{D,k} \sum_{\ell=1}^{J_{k|k-1}} w_{k|k-1}^{(\ell)} \mathcal{N}(z; \hat{z}_{k|k-1}^{(\ell)}, S_{k|k-1}^{(\ell)})} \tag{33}$$

$$m_{k|k}^{(i)}(z) = m_{k|k-1}^{(i)} + K_k(z - \hat{z}_{k|k-1}), \tag{34}$$

$$P_{k|k}^{(i)} = [I - K_k H_k] P_{k|k-1}^{(i)}, \tag{35}$$

$$\hat{z}_{k|k-1} = H_k m_{k|k-1} \tag{36}$$

$$K_k = P_{k|k-1} H_k^T S_{k|k-1}^{-1} \tag{37}$$

$$S_{k|k-1} = H_k P_{k|k-1} H_k^T + R_k. \tag{38}$$

In in a similar manner to the prediction step, the updated means and covariances can be calculated using the Kalman, extended Kalman or unscented Kalman update step. We show that this step can be replaced with importance sampling.

The Gaussian Mixture Particle PHD Filter

The Gaussian Mixture Particle PHD filter presented here represents the Gaussian components in the Gaussian mixture PHD filter [10] with Gaussian particle filters [13] in a similar way to the Gaussian sum particle filter [14]. This has the advantage over the Particle PHD filter that clustering is not required and has a lower complexity since resampling is not required. It has the advantage over the Gaussian mixture PHD filter in that non-linear models can be incorporated. A pseudo-code version of the algorithm is given in Table I.

The single target transition density, $f_{k|k-1}(x_k|x_{k-1})$, and single target likelihood function, $g_k(z_k|x_k)$, no longer need to be linear as in the case of the Gaussian mixture PHD filter. (Should either of these be linear, the equations presented here can be replaced with the those of the GM-PHD filter.)

The class of non-linear multi-target models consists of non-linear transition and observation models for individual targets. Assumption **A4** is relaxed to allow for more general dynamics. In particular, we assume the following.

A.7. Each target follows a non-linear Gaussian dynamic model and non-linear Gaussian measurement model with

$$f_{k|k-1}(x|\zeta) = \mathcal{N}(x; \varphi_{k-1}(\zeta), Q_{k-1}), \tag{39}$$

$$g_k(z|x) = \mathcal{N}(z; h_k(x), R_k), \tag{40}$$

where $\mathcal{N}(\cdot; m, P)$ denotes a Gaussian density with mean m and covariance P, φ_{k-1} is the state transition function, Q_{k-1} is the process noise covariance, h_k is the observation function, and R_k is the observation noise covariance.

GMP-PHD Prediction: Suppose that the posterior distribution at time $k-1$ is given by a mixture of Gaussians of the form (58), Substituting the posterior intensity v_{k-1}, non-linear transition density $f_{k|k-1}$ and birth intensity γ_k into the prediction equation for the PHD, we have

$$\begin{aligned} v_{k|k-1}(x) &= p_{S,k} \int \mathcal{N}(x; \varphi_{k-1}(\xi), Q_{k-1}) \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)} \mathcal{N}(\xi; m_{k-1}^{(i)}, P_{k-1}^{(i)}) d\xi \\ &+ \gamma_k(x) \end{aligned} \tag{41}$$

$$\begin{aligned} &= p_{S,k} \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)} \int \mathcal{N}(x; \varphi_{k-1}(\xi), Q_{k-1}) \mathcal{N}(\xi; m_{k-1}^{(i)}, P_{k-1}^{(i)}) d\xi \\ &+ \gamma_k(x), \end{aligned} \tag{42}$$

where it is assumed that the spontaneous birth intensity is a mixture of Gaussians of the form (22) The integrals in the summation above for each prediction component i, no longer have a closed form solution since it is assumed that $f_{k|k-1}(x|\xi)$ is a non-linear Gaussian transition. These can be calculated using Monte Carlo integration as follows.

For each of the prediction components, $i = 1 \ldots J_{k-1}$, and $j = 1, \ldots, M$, sample particles $x_{k-1}^{(i)(j)}$ from $\mathcal{N}(\cdot; m_{k-1}^{(i)}, P_{k-1}^{(i)})$ and $x_{S,k}^{(i)(j)}$ from $f_{k|k-1}(\cdot|x_{k-1}^{(i)(j)})$. The integral can be approximated by

$$\frac{1}{M} \sum_{i=1}^{M} f_{k|k-1}(x|x_{k-1}^{(i)(j)}), \tag{43}$$

which, by the strong law of large numbers, converges almost surely to

$$\int f_{k|k-1}(x|\xi) \mathcal{N}(\xi; m_{k-1}^{(i)}, P_{k-1}^{(i)}) d\xi, \tag{44}$$

as $M \to \infty$. By taking the sample means and covariances for each component,

$$m_{S,k|k-1}^{(i)} = \frac{1}{M} \sum_{j=1}^{M} x_{S,k|k-1}^{(i)(j)} \tag{45}$$

$$P_{S,k|k-1}^{(i)} = \frac{1}{M} \sum_{j=1}^{M} \left(m_{S,k|k-1}^{(i)} - x_{S,k|k-1}^{(i)(j)} \right) \left(m_{S,k|k-1}^{(i)} - x_{S,k|k-1}^{(i)(j)} \right)^T \tag{46}$$

the predicted intensity from existing targets can be approximated with

$$v_{S,k|k-1}(x) = p_{S,k} \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)} \mathcal{N}(x; m_{S,k|k-1}^{(i)}, P_{S,k|k-1}^{(i)}). \tag{47}$$

Kotecha and Djuric [13] showed that the means and covariances converge to the MMSE of the means and covariances.

GMP-PHD Update: Suppose that the prediction distribution to time k is given by a mixture of Gaussians of the form (64) Substituting $v_{k|k-1}$, non-linear observation g_k into the PHD update, we have

$$v_k(x) = (1-p_{D,k})v_{k|k-1}(x) + \sum_{z\in Z_k}\sum_{i=1}^{J_{k|k-1}} \frac{p_{D,k}w_{k|k-1}^{(i)}\mathcal{N}(z;h_k(x),R_k)\mathcal{N}(x;m_{k|k-1}^{(i)},P_{k|k-1}^{(i)})}{\kappa_k(z)+p_{D,k}\sum_{l=1}^{J_{k|k-1}} w_{k|k-1}^{(l)}W_k^{(l)}(z)}. \quad (48)$$

where

$$W_k^{(l)}(z) = \int \mathcal{N}(z;h_k(\xi),R_k)\mathcal{N}(\xi;m_{k|k-1}^{(l)},P_{k|k-1}^{(l)})d\xi \quad (49)$$

Since the transition functions are non-linear, the properties of Gaussians used to provide the closed-form solution are no longer valid. Importance sampling is performed for each of the measurements and each Gaussian prediction component. For $i=1,\ldots,J_{k-1}$ and for each $z\in Z_k$, draw sample particles $x_k^{(i)(j)}$ from an importance function $\pi_k^{(i)}(\cdot|Z_{1:k-1},z)$ where $j=1,\ldots,M$. Compute the weight $\xi_k^{(i)(j)}(z)$ of each sample $x_k^{(i)(j)}$ according to

$$\xi_k^{(i)(j)}(z) = \frac{\mathcal{N}(z;h_k(x_k^{(i)(j)}),R_k)\mathcal{N}(x_k^{(i)(j)};m_{k|k-1}^{(i)},P_{k|k-1}^{(i)})}{\pi_k^{(i)}\left(x_k^{(i)(j)}|Z_{1:k-1},z\right)}.$$

(Choices for this importance function could include the associated prediction component, $\mathcal{N}(x_{k|k-1};m_{k|k-1}^{(i)},P_{k|k-1}^{(i)})$, or Kalman/ EKF/ UKF measurement updated Gaussian with z_k.)

The sum of these weights converges to the integral in the denominator of the PHD update above, which can be shown as follows

$$\frac{1}{M}\sum_{j=1}^{M}\xi_k^{(i)(j)}(z) \quad (50)$$

$$= \sum_{j=1}^{M}\frac{\mathcal{N}(z;h_k(x_k^{(i)(j)}),R_k)\mathcal{N}(x_k^{(i)(j)};m_{k|k-1}^{(i)},P_{k|k-1}^{(i)})}{\pi_k^{(i)}\left(x_k^{(i)(j)}|Z_{1:k-1},z\right)}$$

$$\to \int \frac{\mathcal{N}(z;h_k(x_k^{(i)}),R_k)\mathcal{N}(x_k^{(i)};m_{k|k-1}^{(i)},P_{k|k-1}^{(i)})}{\pi_k^{(i)}\left(x_k^{(i)}|Z_{1:k-1},z\right)}\times \pi_k^{(i)}\left(x_k^{(i)}|Z_{1:k-1},z\right)dx_k^{(i)} \quad (51)$$

$$= \int \mathcal{N}(z;h_k(x_k^{(i)}),R_k)\mathcal{N}(x_k^{(i)};m_{k|k-1}^{(i)},P_{k|k-1}^{(i)})dx_k^{(i)} \quad (52)$$

Then the Gaussian non-linear PHD update is approximated with

$$v_k(x) = (1-p_{D,k})v_{k|k-1}(x) + \sum_{z\in Z_k}\sum_{j=1}^{J_{k|k-1}} w_k^{(j)}(z)\mathcal{N}(x;m_{k|k}^{(j)}(z),P_{k|k}^{(j)}), \quad (53)$$

where the means, $m_k^{(i)}(z_k)$, and covariances, $P_k^{(i)}$, are approximated with the Gaussian particle update

$$m_k^{(i)} = \frac{\sum_{j=1}^{M}\xi_k^{(i)(j)}(z_k)x_k^{(i)(j)}}{\sum_{j=1}^{M}\xi_k^{(i)(j)}(z_k)}, \quad (54)$$

$$P_k^{(i)} = \frac{\sum_{j=1}^{M}\xi_k^{(i)(j)}(z_k)(m_k^{(i)}-x_k^{(i)(j)})(m_k^{(i)}-x_k^{(i)(j)})^T}{\sum_{j=1}^{M}\xi_k^{(i)(j)}(z_k)}, \quad (55)$$

and weights are calculated with

$$w_k^{(i)}(z) = \frac{p_{D,k}w_{k|k-1}^{(i)}\frac{1}{M}\sum_{j=1}^{M}\xi_k^{(i)(j)}(z)}{\kappa_k(z)+p_{D,k}\sum_{l=1}^{J_{k|k-1}} w_{k|k-1}^{(\ell)}\frac{1}{M}\sum_{j=1}^{M}\xi_k^{(\ell)(j)}(z)}. \quad (56)$$

The sum in the numerator comes from the following approximation of the Gaussian property in (24)

$$\int \mathcal{N}(z;h_k(\xi),R_k)\mathcal{N}(\xi;m_{k|k-1}^{(i)},P_{k|k-1}^{(i)})d\xi\mathcal{N}(x;m_k(z)^{(i)},P_k) \quad (57)$$

$$\approx \mathcal{N}(z;h_k(x_k^{(i)}),R_k)\mathcal{N}(x_k^{(i)};m_{k|k-1}^{(i)},P_{k|k-1}^{(i)})$$

A pseudo-code implementation of the algorithm is presented in Table I, which describes the prediction and update steps in the algorithm. The complexity of the algorithm is controlled by pruning components with low weights and merging components which are close together. Target states can be estimated from the mixture by taking the components with the highest weights. Further details of these procedures can be found in [10].

The Gaussian Mixture Particle CPHD Filter

The Gaussian particle techniques described in the previous section can be also be extended to the CPHD recursions (8)-(13). The key idea and rationale behind the Gaussian particle approach are the same as that for the PHD recursion. The Gaussian Mixture Particle CPHD recursions are outlined as follows.

GMP-CPHD Prediction: Suppose at time $k-1$ that the posterior intensity v_{k-1} and posterior cardinality p_{k-1} are given, and that v_{k-1} is a Gaussian mixture of the form

$$v_{k-1}(x) = \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)}\mathcal{N}(x;m_{k-1}^{(i)},P_{k-1}^{(i)}). \quad (58)$$

For $i=1,\ldots,J_{k-1}$ draw sample particles $x_{k-1}^{(i)(j)}$ from $\mathcal{N}(\cdot;m_{k-1}^{(i)},P_{k-1}^{(i)})$ where $j=1,\ldots,M$. For $i=1,\ldots,J_{k-1}$ and $j=1,\ldots,M$ sample $x_{S,k|k-1}^{(i)(j)}$ from $f_{k|k-1}(\cdot|x_{k-1}^{(i)(j)})$.

Then, the CPHD prediction is approximated by

$$p_{k|k-1}(n) = \sum_{j=0}^{n} p_{\Gamma,k}(n-j) \sum_{\ell=j}^{\infty} C_j^\ell p_{k-1}(\ell) p_{S,k}^j (1-p_{S,k})^{\ell-j} \quad (59)$$

$$v_{k|k-1}(x) = v_{S,k|k-1}(x) + \gamma_k(x), \quad (60)$$

where $\gamma_k(x)$ is given in (22),

$$v_{S,k|k-1}(x) = p_{S,k} \sum_{i=1}^{J_{k-1}} w_{k-1}^{(i)} \mathcal{N}(x; m_{S,k|k-1}^{(i)}, P_{S,k|k-1}^{(i)}), \quad (61)$$

$$m_{S,k|k-1}^{(i)} = \frac{1}{M} \sum_{j=1}^{M} x_{S,k|k-1}^{(i)(j)} \quad (62)$$

$$P_{S,k|k-1}^{(i)} = \frac{1}{M} \sum_{j=1}^{M} \left(m_{S,k|k-1}^{(i)} - x_{S,k|k-1}^{(i)(j)} \right) \left(m_{S,k|k-1}^{(i)} - x_{S,k|k-1}^{(i)(j)} \right)^T \quad (63)$$

GMP-CPHD Update: Suppose at time $k-1$ that the predicted intensity $v_{k|k-1}$ and predicted cardinality $p_{k|k-1}$ are given, and that $v_{k|k-1}$ is a Gaussian mixture of the form

$$v_{k|k-1}(x) = \sum_{i=1}^{J_{k|k-1}} w_{k|k-1}^{(i)} \mathcal{N}(x; m_{k|k-1}^{(i)}, P_{k|k-1}^{(i)}). \quad (64)$$

For $i = 1, ..., J_{k-1}$ and for each $z \in Z_k$, draw sample particles $x_k^{(i)(j)}$ from an importance function $\pi_k^{(i)}(\cdot|Z_{1:k-1}, z)$ where $j = 1, ..., M$.

Compute the weight $\xi_k^{(i)(j)}(z)$ of each sample $x_k^{(i)(j)}$ according to

$$\xi_k^{(i)(j)}(z) = \frac{g_k(z|x_k^{(i)(j)}) \mathcal{N}(x_k^{(i)(j)}; m_{k|k-1}^{(i)}, P_{k|k-1}^{(i)})}{\pi_k^{(i)} \left(x_k^{(i)(j)} | Z_{1:k-1}, z \right)}.$$

Then, the CPHD update is approximated by

$$p_k(n) = \frac{\Psi_k^0[w_{k|k-1}, Z_k](n) p_{k|k-1}(n)}{\langle \Psi_k^0[w_{k|k-1}, Z_k], p_{k|k-1} \rangle} \quad (65)$$

$$v_k(x) = (1-p_{D,k}) \frac{\langle \Psi_k^1[w_{k|k-1}, Z_k], p_{k|k-1} \rangle}{\langle \Psi_k^0[w_{k|k-1}, Z_k], p_{k|k-1} \rangle} v_{k|k-1}(x) + \sum_{z \in Z_k} \sum_{i=1}^{J_{k|k-1}} w_k^{(i)}(z) \mathcal{N}(x; m_k^{(i)}(z), P_k^{(i)}) \quad (66)$$

where

$$\Psi_k^u[w, Z](n) = \sum_{j=0}^{\min(|Z|,n)} (|Z|-j)! p_{K,k}(|Z|-j) P_{j+u}^n \times \frac{(1-p_{D,k})^{n-(j+u)}}{\langle 1, w \rangle^{j+u}} e_j(\Xi_k(w, Z)), \quad (67)$$

$$\Xi_k(w, Z) = \left\{ \frac{\langle 1, \kappa_k \rangle}{\kappa_k(z)} p_{D,k} w^T \xi_k(z) : z \in Z \right\}, \quad (68)$$

$$w_{k|k-1} = [w_{k|k-1}^{(1)}, ..., w_{k|k-1}^{(J_{k|k-1})}]^T, \quad (69)$$

$$\xi_k(z) = [\xi_k^{(1)}(z), ..., \xi_k^{(J_{k|k-1})}(z)]^T, \quad (70)$$

$$\xi_k^{(i)}(z) = \frac{1}{M} \sum_{j=1}^{M} \xi_k^{(i)(j)}(z), \quad (71)$$

$$w_k^{(i)}(z) = p_{D,k} w_{k|k-1}^{(i)} \xi_k^{(i)}(z) \frac{\langle \Psi_k^1[w_{k|k-1}, Z_k \backslash \{z\}], p_{k|k-1} \rangle}{\langle \Psi_k^0[w_{k|k-1}, Z_k], p_{k|k-1} \rangle} \times \frac{\langle 1, \kappa_k \rangle}{\kappa_k(z)}, \quad (72)$$

$$m_k^{(i)}(z) = \frac{\sum_{j=1}^{M} \xi_k^{(i)(j)}(z) x_k^{(i)(j)}}{\sum_{j=1}^{M} \xi_k^{(i)(j)}(z)}, \quad (73)$$

$$P_k^{(j)} = \frac{\sum_{j=1}^{M} \xi_k^{(i)(j)}(z) \left(m_k^{(i)}(z) - x_k^{(i)(j)} \right) \left(m_k^{(i)}(z) - x_k^{(i)(j)} \right)^T}{\sum_{j=1}^{M} \xi_k^{(i)(j)}(z)}. \quad (74)$$

4. Simulation Results

In this example, the non-linear tracking performance of the GMP-PHD and GMP-CPHD filters is demonstrated. The target motions are modelled on a nearly constant turn model with varying turn rate, and the corresponding observations are modelled as noisy bearing and range measurements. In this simulation, the observation region is the half disc of radius $2000m$ in which a total of 5 targets enter and leave the scene at various times. A plot of the ground truths is shown in Fig 1. A circular dot denotes the start of a track and a square dot denotes the end of a track, whilst the start and stop times for each track are labelled on the plot. The target state variable $x_k = [\ \tilde{x}_k^T, \omega_k\ ]^T$ is comprised of the planar position and velocity $\tilde{x}_k^T = [\ p_{x,k}, \dot{p}_{x,k}, p_{y,k}, \dot{p}_{y,k}\ ]^T$ as well as the turn rate ω_k. The state transition model is

$$\tilde{x}_k = F(\omega_{k-1}) \tilde{x}_{k-1} + G w_{k-1} \quad (75)$$

$$\omega_k = \omega_{k-1} + \Delta u_{k-1} \quad (76)$$

where

$$F(\omega) = \begin{bmatrix} 1 & \frac{\sin \omega\Delta}{\omega} & 0 & -\frac{1-\cos\omega\Delta}{\omega} \\ 0 & \cos\omega\Delta & 0 & -\sin\omega\Delta \\ 0 & \frac{1-\cos\omega\Delta}{\omega} & 1 & \frac{\sin\omega\Delta}{\omega} \\ 0 & \sin\omega\Delta & 0 & \cos\omega\Delta \end{bmatrix}, \quad G = \begin{bmatrix} \frac{\Delta^2}{2} & 0 \\ T & 0 \\ 0 & \frac{\Delta^2}{2} \\ 0 & \Delta \end{bmatrix},$$

$w_{k-1} \sim \mathcal{N}(\cdot; 0, \sigma_w^2 I)$, and $u_{k-1} \sim \mathcal{N}(\cdot; 0, \sigma_u^2 I)$ with $\Delta = 1s$, $\sigma_w = 15m/s^2$, and $\sigma_u = 2\pi/180 rad/s$. The sensor observation is a noisy bearing and range vector given by

$$z_k = \begin{bmatrix} \arctan(p_{x,k}/p_{y,k}) \\ \sqrt{p_{x,k}^2 + p_{y,k}^2} \end{bmatrix} + \varepsilon_k \quad (77)$$

where $\varepsilon_k \sim \mathcal{N}(\cdot; 0, R_k)$, with $R_k = \mathrm{diag}([\ \sigma_\theta^2, \sigma_r^2\ ]^T)$, $\sigma_\theta = 6(\pi/180) rad$, and $\sigma_r = 15m$. The birth process is modelled on a Poisson RFS with intensity $\gamma_k(x) = 0.1\mathcal{N}(x; m_\gamma^{(1)}, P_\gamma) + 0.1\mathcal{N}(x; m_\gamma^{(2)}, P_\gamma)$ where $m_\gamma^{(1)} = [\,-1000, 0, -500, 0\, 0\,]^T, m_\gamma^{(2)} = [\,1050, 0, 1070, 0\, 0\,]^T$, and $P_\gamma = \text{diag}([\,50,\ 50,\ 50,\ 50,\ 6(\pi/180)\,]^{\text{T}})$. The probability of target survival and detection are $p_{S,k} = 0.99$ and $p_{D,k} = 0.98$ respectively. Clutter is modelled on a Poisson RFS with intensity $\lambda_c = 3.98 \times 10^{-3}\ (radm)^{-1}$ over the region $[-\pi/2, \pi/2]rad \times [0, 2000]m$ (giving an average of 25 clutter points per scan).

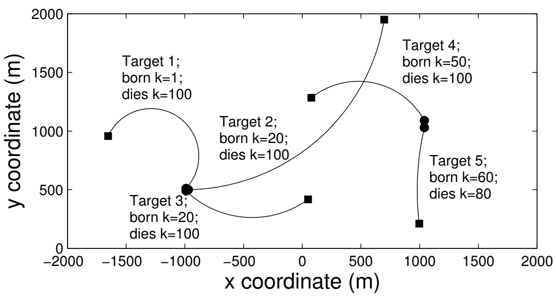

Figure 1. Target trajectories shown in *xy* plane

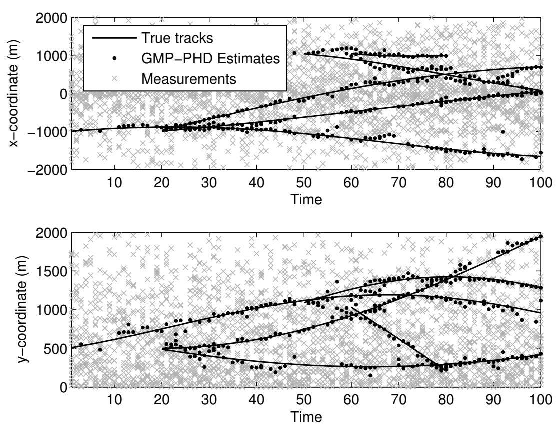

Figure 2. *x-y* coordinates of target estimates of the GMP-PHD filter in time

Both the GMP-PHD and GMP-CPHD filters are given the same measurement data, and both are run with 100 particles using the predicted mixture components as proposal distributions. To mitigate the exponential growth of mixture components, at each time step the number of Gaussian components is capped to a maximum of $J_{max} = 100$ components, whilst pruning is performed with a weight threshold of $T = 10^{-5}$, and merging is performed with a threshold of $U = 4m$. For the GMP-CPHD filter, the cardinality distribution is calculated to $N_{\max} = 100$ terms. The true trajectories and filter outputs are shown in x and y coordinates versus time for the GMP-PHD and GMP-CPHD filters in Fig. 2 and Fig. 3 respectively. These preliminary results suggest that the GMP-PHD/CPHD filters are suitable for tracking non-linear target motions in the presence of births, deaths, clutter and missed detections. It is also observed that both filters have no trouble resolving the two targets which cross paths at time $k = 80$. In addition, the true and estimated number of targets versus time is shown for both filters in Fig 4. As expected, it is observed that the CPHD based filter produces more steady and accurate cardinality estimates compared to its PHD based counterpart, although at the expense of added computational cost.

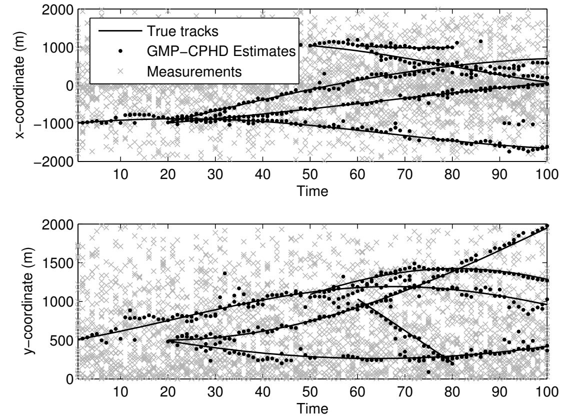

Figure 3. *x-y* coordinates of target estimates of the GMP-CPHD filter in time

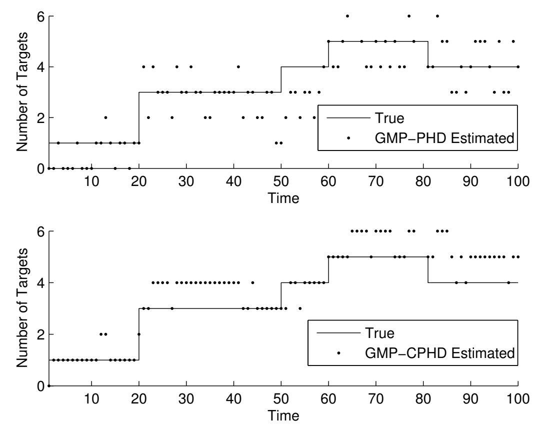

Figure 4. True and estimated target numbers for the GMP-PHD/CPHD filters in time

5. Concluding Remarks

In this paper we have presented some preliminary results on Gaussian mixture particle implementations of the PHD and CPHD recursions. The Gaussian mixture particle approach does not require clustering for extracting state estimates and can be applied to highly non-linear problems. Simulations showed that the approach yields encouraging tracking performance. However, preliminary numerical studies revealed a numerical problem with our current implementations. Occasionally the algorithm breaks down due to the covariance matrices of the Gaussian components collapsing to singularities. At this stage we do not know what causes this to happen. Further studies are being carried out to rectify this problem.

Table 1. Pseudo-code for the Gaussian Mixture Particle PHD filter.

---

given $\{w_{k-1}^{(i)}, m_{k-1}^{(i)}, P_{k-1}^{(i)}\}_{i=1}^{J_{k-1}}$, and the measurement set Z_k.

step 1. (prediction for birth targets)

$i = 0.$

for $j = 1, \dots, J_{\gamma,k}$

$i := i+1.$

$w_{k|k-1}^{(i)} = w_{\gamma,k}^{(j)}, \quad m_{k|k-1}^{(i)} = m_{\gamma,k}^{(j)}, \quad P_{k|k-1}^{(i)} = P_{\gamma,k}^{(j)}.$

end

step 2. (prediction for existing targets)

for $j = 1, \dots, J_{k-1}$

$i := i+1.$

for $j = 1, \dots, M$

sample $x_{k-1}^{(j)(\ell)} \sim \mathcal{N}(\cdot; m_{k-1}^{(j)}, P_{k-1}^{(j)})$

predict $x_{S,k|k-1}^{(i)(\ell)} = \varphi_{k-1}(x_{k-1}^{(j)(\ell)})$

end

$w_{k|k-1}^{(i)} = p_{S,k}\, w_{k-1}^{(j)}.$

$m_{k|k-1}^{(i)} = \frac{1}{M}\sum_{\ell=1}^{M} x_{S,k|k-1}^{(i)(\ell)}$

$P_{k|k-1}^{(i)} = \frac{1}{M}\sum_{\ell=1}^{M}\left(m_{k|k-1}^{(i)} - x_{S,k|k-1}^{(i)(\ell)}\right)\left(m_{k|k-1}^{(i)} - x_{S,k|k-1}^{(i)(\ell)}\right)^T,$

end

$J_{k|k-1} = i.$

step 3. (update)

for $j = 1, \dots, J_{k|k-1}$

$w_k^{(j)} = (1 - p_{D,k}) w_{k|k-1}^{(j)}, \quad m_k^{(j)} = m_{k|k-1}^{(j)}, \quad P_k^{(j)} = P_{k|k-1}^{(j)}.$

end

$\ell := 0.$

for each $z \in Z_k$

$\ell := \ell + 1.$

for $j = 1, \dots, J_{k|k-1}$

for $n = 1 \dots M$

sample $x_k^{(j)(n)} \sim \pi_k(\cdot | Z_{1:k-1}, z)$

calculate particle weight

$\xi_k^{(j)(n)}(z) = \frac{\mathcal{N}(z; h_k(x_k^{(j)(n)}), R_k)\,\mathcal{N}(x_k^{(j)(n)}; m_{k|k-1}^{(j)}, P_{k|k-1}^{(j)})}{\pi_k^{(i)}\left(x_k^{(j)(n)} | Z_{1:k-1}, z\right)}$

end

$w_k^{(\ell J_{k|k-1}+j)} = p_{D,k}\, w_{k|k-1}^{(j)} \frac{1}{M}\sum_{n=1}^{M} \xi_k^{(j)(n)}(z).$

$m_k^{(\ell J_{k|k-1}+j)} = \frac{\sum_{n=1}^{M} \xi_k^{(j)(n)}(z) x_k^{(j)(n)}}{\sum_{n=1}^{M} \xi_k^{(j)(n)}(z)},$

$P_k^{(\ell J_{k|k-1}+j)} = \frac{\sum_{n=1}^{M} \xi_k^{(j)(n)}(z)\left(m_k^{(\ell J_{k|k-1}+j)} - x_k^{(j)(n)}\right)\left(m_k^{(\ell J_{k|k-1}+j)} - x_k^{(j)(n)}\right)^T}{\sum_{n=1}^{M} \xi_k^{(j)(n)}(z)}$

end

$w_k^{(\ell J_{k|k-1}+j)} := \frac{w_k^{(\ell J_{k|k-1}+j)}}{\kappa_k(z) + \sum_{i=1}^{J_{k|k-1}} w_k^{(\ell J_{k|k-1}+i)}}$, for $j = 1, \dots, J_{k|k-1}$.

end

$J_k = \ell J_{k|k-1} + J_{k|k-1}.$

output $\{w_k^{(i)}, m_k^{(i)}, P_k^{(i)}\}_{i=1}^{J_k}.$

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